Indian Statistical Institute, Bangalore
B. Math.(hons.), Third Year, First Semester
Probability-III

End Term Examination — Date : 12 November 2024
Maximum marks: 50 — Time: 3 hours

Answer any 5, each question carries 10 marks.

Kindly ensure your writing is clear and if you are using any results please provide as many details as you can.

1. (a) Suppose $\mathcal{F}$ is a sigma-algebra on X. Determine if each of the following statements is true or false, and explain your reasoning. [**5**]
   i. Let $T : X \to Y$ be a function. The collection $\mathcal{G}_1 := \{T(A) : A \in \mathcal{F}\}$ forms a sigma-algebra on Y.
   ii. Let $T : Y \to X$ be a function. The collection $\mathcal{G}_2 := \{T^{-1}(A) : A \in \mathcal{F}\}$ is a sigma-algebra on Y, where $T^{-1}(A) = \{y \in Y : T(y) \in A\}$.
   iii. It is impossible to have exactly 101 elements in any sigma-algebra.

   (b) Let τ be a topology on a set X, and let $A = A(\tau)$ be the algebra generated by τ. Show that A consists of subsets of X that can be expressed as finite unions of sets of the form $F \cap V$, where F is closed and V is open. [**5**]

2. (a) Let Y be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$,be a sub-sigma algebra of $\mathcal{F}$. If $\mathbb{E}Y^2 < \infty$, then show that $Var(Y) = Var(\mathbb{E}(Y|\mathcal{G})) + \mathbb{E}(Var(Y|\mathcal{G}))$. [**5**]

   (b) Let X be an Exponential(1) random variable. For $t > 0$, let $Y_1 = \min\{X, t\}$ and $Y_2 = \max\{X, t\}$. Find $\mathbb{E}(X \mid Y_i)$ for $i = 1, 2$. [**5**]

3. (a) Let $0 \leq X_1 \leq X_2 \leq \dots$ be random variables with $\mathbb{E}[X_n] \sim an^{\alpha}$ for some $a, \alpha > 0$, and $\mathrm{Var}(X_n) \leq bn^{\beta}$ where $\beta < 2\alpha$ and b is a finite constant. Then, show that $\frac{X_n}{n^{\alpha}} \to a$ a.s. [**5**]

   (b) Let X_n be independent Poisson random variables with $\mathbb{E}[X_n] = \lambda_n$, and let $S_n = X_1 + \cdots + X_n$. Show that if $\sum_{n=1}^{\infty} \lambda_n = \infty$, then $\frac{S_n}{\mathbb{E}[S_n]} \to 1$ a.s. [**5**]

4. (a) Flip a fair coin 200 times. Estimate **(i)** the probability of getting more than 60 heads, **(ii)** the probability of getting between 50 and 80 heads. [**4**]

   (b) Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that for $n \geq 1$, $X_n \sim \mathrm{Poisson}(\lambda_n)$, where $\lambda_n \in (0, \infty)$. Define $Y_n = \frac{X_n - \lambda_n}{\sqrt{\lambda_n}}$ for $n \geq 1$. If $\lambda_n \to \infty$ as $n \to \infty$, then $Y_n \xrightarrow{d} N(0, 1)$. [**6**]

5. Let $\mu_n \in Prob(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and let μ be a finite Borel measure on $\mathbb{R}$. Let $F_n(t) = \mu_n(-\infty, t]$ and $F(t) = \mu(-\infty, t]$. Then $\mu_n \to \mu$ vaguely if and only if $F_n(b) - F_n(a) \to F(b) - F(a), \quad \forall a, b \in Cont(F)$. **[10]**

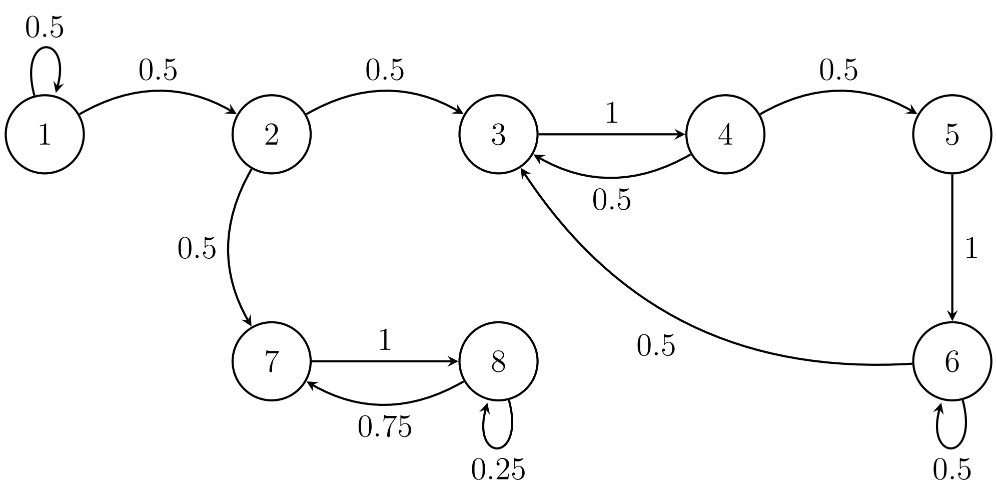
6. (a) Suppose that μ, ν, and γ are probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Show that $d_{TV}(\mu * \nu, \mu * \gamma) \leq d_{TV}(\nu, \gamma)$, where $d_{TV}(\mu, \nu) := \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|$ is the total variation distance. Using this fact show that

$$d_{TV}(\mu_1 * \mu_2 * \cdots * \mu_n, \nu_1 * \nu_2 * \cdots \nu_n) \leq \sum_{i=1}^{n} d_{TV}(\mu_i, \nu_i)$$

for all choices of probability measures μ_i and ν_i on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. **[5]**

(b) Let $\{Z_i\}_{i=1}^n$ be independent Bernoulli random variables with $P(Z_i = 1) = p_i \in (0, 1)$ and $P(Z_i = 0) = 1 - p_i$. Define $S := Z_1 + \cdots + Z_n, \quad a := p_1 + \cdots + p_n, \quad$ and $\quad X \sim \text{Poi}(a)$. Then, $d_{TV}(S_n, X) \leq \sum_{i=1}^{n} p_i^2$. **[5]**

7. Consider the following transition diagram of a Markov Chain $\{X_n\}_{n \geq 0}$, on state space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$



Then,

(a) Write the transition matrix and identify the classes.

(b) Determine the transient and recurrent states and then write the canonical form of transition matrix.

[5+5]